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A CHARACTERIZATION OF CONNECTED (WEAKLY) ORDERABLE SPACES

BY

A.E. BROUWER

Introduction

A topological space X is said to be (weakly) orderable if there exists a total order $<$ on X such that all open intervals $(a,b) = \{x \in X \mid a < x < b\}$ are open in X.
In the following X will denote a connected T_1-space.
X is said to have the property

(B') iff for each $p \in X$ $X \setminus p$ has at most two components.

(B'0) iff for each $p \in X$ all components of $X \setminus p$ are open.

Clearly (B') implies (B'0).
It is well known (see e.g. [2] and [3]) that a connected topological space X is orderable iff among every three points of X there is exactly one which separates the other two.
Here we prove:

Theorem

If the connected T_1-space X satisfies the following three conditions:

(i) among any three points of X there is at least one which lies in a connected set separating the other two;

(ii) among any three points of X there is at most one which lies in an open connected set which separates the other two points;

(iii) (B'0),

then X is orderable, (and conversely, an orderable space certainly satisfies (i) - (iii)).

Corollary 1

If among three points of X there is exactly one which has an open connected neighbourhood which separates the other two then X is orderable.

Corollary 2

If among any three points of X there is exactly one which lies in a connected set that separates the other two, then X is orderable.

We use the following notation:

$$A = \underset{b}{B} + \underset{c}{C}$$

means that A is the topological sum of its subsets B and C, $b \in B$ and $c \in C$.

Proof of the corollaries

In both cases it is sufficient to prove that X satisfies B' since B' implies B'0. Suppose $X \setminus p = \underset{a}{A} + \underset{b}{B} + \underset{c}{C}$.

Then $A \cup \{p\}$, $B \cup \{p\}$ and $C \cup \{p\}$ are connected sets containing a resp. b resp. c and separating b and c resp. a and c resp. a and b which gives a contradiction in case 2. In case 1 let U_{ab} be an open connected neigbourhood of p which does not contain a and b, then $C \cup U_{ab}$ is an open connected neighbourhood of c separating a and b. Likewise $B \cup U_{ac}$ and $A \cup U_{bc}$ are open connected neighbourhoods of b resp. a separating a and c resp. b and c. Contradiction.

Proof of the theorem

1. X satisfies B'.
   For, suppose $X \setminus p = \underset{a}{A} + \underset{b}{B} + \underset{c}{C}$; $\bar{A} \setminus a = \underset{p}{A_1} + A_2$;

   $\bar{B} \setminus b = \underset{p}{B_1} + B_2$; $\bar{C} \setminus c = \underset{p}{C_1} + C_2$ (where A_2, B_2, C_2 may be empty).

Since the components of $X \setminus a$ are open, the components of $\bar{A} \setminus a$ are open in $\bar{A}$, hence we may assume that A_1, B_1 and C_1 are connected. Now $A \cup B_1 \cup C_1$ is an open connected neighbourhood of a separating b and c, and $B \cup A_1 \cup C_1$ is an open connected neighbourhood of b separating a and c, a contradiction.

2. The complement of an open connected set has at most two components. For, suppose $X \setminus C = \underset{p_1}{A_1} + \underset{p_2}{A_2} + \underset{p_3}{A_3}$ where C is open and connected.

   Then $C \cup A_i$ is an open connected neighbourhood of p_i separating p_j and p_k ($i = 1,2,3$; $i \neq j \neq k \neq i$).
   Contradiction.

3. If X contains at least two cut points, X is orderable. For, let p and q be two cut points; $X \setminus p = \underset{a}{A_p} + \underset{q}{B_p}$, $X \setminus q = \underset{p}{A_q} + \underset{b}{B_q}$.

   Then $\overline{A_p} \subset A_q$, and so $a \in A_q$ and $b \in B_p$.
   $Y = X \setminus (A_p \cup B_q) = \overline{A_q} \setminus A_p$ is closed and connected;
   $Y^\circ = Y \setminus \{p,q\}$.
   Let $\overline{A_p} \setminus a = \underset{p}{E_a} + F_a$ and $\overline{B_q} \setminus b = \underset{q}{E_b} + F_b$ where E_a and E_b are connected and F_a and F_b may be empty. Then $Y \cup E_a \cup E_b$ is connected and open.

3A. Y can have no endpoints other than p and q.
   For if $Y \setminus r$ is connected ($r \neq p,q$), $(Y \setminus r) \cup E_a \cup E_b$ is open and connected with complement $\{r\} + (F_a \cup \{a\}) + (F_b \cup \{b\})$ which contradicts 2.

3B. Y satisfies B'.
   For if $Y \setminus r = \underset{p}{A} + \underset{q}{B} + C$ or $Y \setminus r = \underset{p,q}{A} + B + C$ then

   $X \setminus r = (A \cup A_p) + (B \cup B_q) + C$ or $X \setminus r = (A \cup A_p \cup A_q) + B + C$ which contradicts 1.

3C. Each point r of Y° separates p and q.
   For if $Y \setminus r = \underset{p,q}{A} + B$ then A and B are connected by 3B, so

$A \cup E_a \cup E_b$ is open and connected and has complement $(F_a \cup \{a\}) + (F_b \cup \{b\}) + (B \cup \{r\})$, which contradicts 2.

3D. Y is orderable.

For Y is connected and $Y = p + E(p,q) + q$ (see Whyburn, [4]).

3E. X is orderable:

Let Z be the collection of all cut points of X.

Then first of all Z is orderable. For, if $z \in A_p$ let $X \setminus z = A_{z_p} + B_{z_p}$ and if $z \in B_p$ let $X \setminus z = A_{z_p} + B_z$.

Then $y < z \leftrightarrow \overline{A_y} \subset A_z$ defines an order on Z, compatible with the topology. Now $X = (\bigcap_{z \in Z} A_z) \cup Z \cup (\bigcap_{z \in Z} B_z)$ since $x \in A_r \cap B_s$ for some $r, s \in Z$ implies that x separates r and s (3C) and therefore that $x \in Z$.

Suppose $\bigcap_{z \in Z} A_z$ contains two distinct points e, f.

Then $\overline{A_p} \setminus e$ is connected, hence $A_q \setminus e$ is an open connected neighbourhood of f separating e and q.

Likewise $A_q \setminus f$ is an open connected neighbourhood of e separating f and q. Contradiction.

Therefore both $\bigcap_{z \in Z} A_z$ and $\bigcap_{z \in Z} B_z$ can contain at most one point, and we can extend the order on Z in the obvious way to an order on X.

This proves 3.

4. X cannot contain exactly one cut point.

Let p be the only cut point of X, and $X \setminus p = A_{p_1,p_2} + B_{p_3}$.

Then $\overline{A} \setminus p_1$ and $\overline{A} \setminus p_2$ and $\overline{B} \setminus p_3$ are connected and consequently $(B \setminus p_3) \cup \{p\} \cup (A \setminus p_i)$ is an open connected neighbourhood of p_j separating p_i and p_3 ($i = 1,2$; $i \neq j$).

Contradiction.

5. X contains at least one cut point.

For suppose no point of X is a cut point. We consider two cases.

5A. Suppose $X \setminus \{p,q\}$ is disconnected for all $p,q \in X$.

Now $X \setminus \{p,q\} = A \underset{r}{+} B$ and $\overline{B} = B \cup \{p,q\}$ is connected. So r cannot lie in a connected set separating p and q. Since p, q, r are arbitrary we arrive at a contradiction.

5B. Let $X \setminus \{p,q\}$ be connected for some fixed pair of distinct points p, q.

Now $X \setminus \{p,q\}$ is an open connected neighbourhood of r separating p and q for each point $r \in X \setminus \{p,q\}$.

Therefore q cannot have an open connected neighbourhood separating p and some other point r.

Hence $X \setminus \{p,r\}$ cannot be connected for $r \neq q$.

Thus $X \setminus p$ has at most one end point.

Choose r_1, r_2 different from p, q.

Let $X \setminus \{p,r_1\} = A_1 \underset{r_2}{+} A_2$ and $X \setminus \{p,r_2\} = B_1 \underset{r_1}{+} B_2$.

Then $\overline{A_1} = A_1 \cup \{p,r_1\}$ and $\overline{B_1} = B_1 \cup \{p,r_2\}$ so r_2 and r_1 cannot lie in a connected set separating r_1 and p resp. r_2 and p. Therefore p must lie in a connected set separating r_1 and r_2 and $X \setminus \{r_1,r_2\}$ must be connected. (Otherwise we would have $X \setminus \{r_1,r_2\} = A \underset{p}{+} B$ and $\overline{B} = B \cup \{r_1,r_2\}$ which gives a contradiction.)

But now in $X \setminus r_1$ all points except possibly p and q are end points, which is impossible by the above argument.

Contradiction.

Finally 3, 4 and 5 together prove the theorem.

Remark

The condition B'0 in the theorem is needed to ensure the existence of sufficiently many open connected sets without which the second condition (among three points there is at most one which lies in an open connected set that separates the other two points) would be useless.

For example, if a V1-space is defined as a connected T_1-space satisfying the following property: "each connected subset has at most one end point" then we have for V_1-spaces:

1. Among every three points there is at least one which lies in a connected set that separates the other two points.
2. The complement of an open connected set is connected.
   Therefore no open connected set separates any two points.
3. For all but at most one point $p \in X$ $X \setminus p$ does have infinitely many open components and exactly one closed component.

(see [1]).

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